Last Time: Proved every real, symmetric matrix has real eigenvalues.

↳ End: Saw an example: we were able to diagonalize a matrix "orthogonally". i.e. we found an orthogonal matrix Q for matrix M and diagonal D w/

$$M = QDQ^T \rightsquigarrow Q \text{ orthogonal} \Rightarrow Q^T = Q^{-1}$$

So this is the same equation as $M = PDP^{-1}$.

Observations: ① If M is a matrix and we can express

∗ $M = QDQ^T$ for Q an orthogonal matrix and D a diagonal matrix, then

$$(AB)^T = B^T A^T$$

$$M^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = Q D^T Q^T = QDQ^T = M.$$

Hence if M is orthogonally diagonalizable, then M is symmetric :)

② $M = QDQ^T$ for Q orthogonal and D diagonal, then $Q^T = Q^{-1}$ implies $M = QDQ^{-1}$, so D is a matrix of eigenvalues of M, and the columns of Q form bases for eigenspaces of M. Because Q is orthogonal, $Q^TQ = I$, so columns of Q are mutually orthogonal; so eigenspaces associated to different e-values are orthogonal!

Point: M orthogonally diagonalizable implies: ① M symmetric ② the eigenspaces of M are mutually orthogonal.

Miraculous: If M is symmetric, then the eigenspaces of M are mutually orthogonal; hence M is orthogonally diag'able.

Ex: $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$P_M(\lambda) = \det(M - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \leftarrow$$

$$= -\lambda \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - 1 \det \begin{bmatrix} 1 & 1 \\ 1 & 1-\lambda \end{bmatrix} + 1 \det \begin{bmatrix} 1 & 1-\lambda \\ 1 & 1 \end{bmatrix}.$$

$$= -\lambda\left((1-\lambda)^2 - 1\right) - \left((1-\lambda) - 1\right) + \left(1 - (1-\lambda)\right)$$

$$= -\underline{\lambda}\left((1-\lambda)^2 - 1\right) - \underline{(-\lambda)} - \underline{(-\lambda)}$$

$$= (-\lambda)\left((1-\lambda)^2 - 1 - 1 - 1\right) = -\lambda\left((1-\lambda)^2 - 3\right)$$

e-values: $P_M(\lambda) = 0$ iff $-\lambda = 0$ OR $(1-\lambda)^2 - 3 = 0$

iff $\lambda = 0$ OR $(1-\lambda)^2 = 3$

iff $\lambda = 0$ OR $1 - \lambda = \pm\sqrt{3}$

iff $\lambda = 0$ OR $\lambda = 1 \pm \sqrt{3}$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{bmatrix}, \quad P = ?$$

verify...

$\lambda_1 = 0$: $V_{\lambda_1} = \text{null}(M - \lambda_1 I) = \text{null}(M) = \text{null}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_1}$ iff $\begin{cases} x = 0 \\ y + z = 0 \end{cases}$ iff $\begin{cases} x = 0 \\ y = -t \\ z = t \end{cases}$

$\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis of the eigenspace V_{λ_1}.

$\lambda_2 = 1 + \sqrt{3}$: $V_{\lambda_2} = \text{null}(M - \lambda_2 I) = \text{null}\begin{bmatrix} -1-\sqrt{3} & 1 & 1 \\ 1 & -\sqrt{3} & 1 \\ 1 & 1 & -\sqrt{3} \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 1 & 1 & -\sqrt{3} \\ 1 & -\sqrt{3} & 1 \\ -1-\sqrt{3} & 1 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -\sqrt{3} \\ 0 & -1-\sqrt{3} & 1+\sqrt{3} \\ 0 & 2+\sqrt{3} & -2-\sqrt{3} \end{bmatrix}$$

$$= \text{null}\begin{bmatrix} 1 & 1 & -\sqrt{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 1-\sqrt{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_2}$ iff $\begin{cases} x + (1-\sqrt{3})z = 0 \\ y - z = 0 \end{cases}$ iff $\begin{cases} x = (1-\sqrt{3})t \\ y = t \\ z = t \end{cases}$ $\therefore B_{\lambda_2} = \left\{ \begin{bmatrix} 1-\sqrt{3} \\ 1 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda_3 = 1-\sqrt{3}}$; $V_{\lambda_3} = \text{null}(M-\lambda_3 I) = \text{null}\begin{bmatrix} -1+\sqrt{3} & 1 & 1 \\ 1 & \sqrt{3} & 1 \\ 1 & 1 & \sqrt{3} \end{bmatrix} = \text{null}\begin{bmatrix} 1 & \sqrt{3} & 1 \\ -1+\sqrt{3} & 1 & 1 \\ 1 & 1 & \sqrt{3} \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 1 & \sqrt{3} & 1 \\ 0 & \sqrt{3}-2 & 2-\sqrt{3} \\ 0 & 1-\sqrt{3} & \sqrt{3}-1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & \sqrt{3} & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \text{null}\begin{bmatrix} 1 & 0 & 1+\sqrt{3} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_3}$ iff $\begin{cases} x + (1+\sqrt{3})z = 0 \\ y - z = 0 \end{cases}$ iff $\begin{cases} x = (1+\sqrt{3})t \\ y = t \\ z = t \end{cases}$ $\therefore \mathcal{B}_{\lambda_3} = \left\{\begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix}\right\}$

$\therefore P = \begin{bmatrix} 0 & 1-\sqrt{3} & 1+\sqrt{3} \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ w/ $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+\sqrt{3} & 0 \\ 0 & 0 & 1-\sqrt{3} \end{bmatrix}$ satisfy

$M = PDP^{-1}$. NB: P is not orthogonal...

in this case, columns are orthogonal, but <u>not</u> orthonormal.

$\left(\text{indeed: } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ 1 \\ 1 \end{bmatrix} = 0 \checkmark \text{ and } \begin{bmatrix} 1-\sqrt{3} \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix} = (1-\sqrt{3})(1+\sqrt{3}) + 1 + 1 = -2+2 = 0\right)$

<u>here we can just scale them:</u>

$\left\|\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\| = \sqrt{0+1+1} = \sqrt{2}$, $\left\|\begin{bmatrix} 1-\sqrt{3} \\ 1 \\ 1 \end{bmatrix}\right\| = \sqrt{(1-\sqrt{3})^2+1+1} = \sqrt{1-2\sqrt{3}+3+2} = \sqrt{6+2\sqrt{3}}$

and $\left\|\begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix}\right\| = \sqrt{(1+\sqrt{3})^2+1+1} = \sqrt{6+2\sqrt{3}}$.

Here $Q = \begin{bmatrix} 0 & \frac{1-\sqrt{3}}{\sqrt{6-2\sqrt{3}}} & \frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}} \\ \frac{-1}{\sqrt{2}} & 1/\sqrt{6-2\sqrt{3}} & 1/\sqrt{6+2\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{6-2\sqrt{3}} & 1/\sqrt{6+2\sqrt{3}} \end{bmatrix}$:) $\square$

NB: We had distinct eigenvalues in this case... what if we didn't?

Ex: $M = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$.

$$p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix}$$

$$= (4-\lambda)\det\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} - 2\det\begin{bmatrix} 2 & 2 \\ 2 & 4-\lambda \end{bmatrix} + 2\det\begin{bmatrix} 2 & 4-\lambda \\ 2 & 2 \end{bmatrix}$$

swap

negate

$$= (4-\lambda)\det\begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} - 4\det\begin{bmatrix} 2 & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

$$= (4-\lambda)\left((4-\lambda)^2 - 2^2\right) - 4\left(2(4-\lambda) - 2\cdot 2\right)$$

$$= (4-\lambda)\underline{(4-\lambda-2)}(4-\lambda+2) - 4\left(2\,\underline{(4-\lambda-2)}\right)$$

$$= (2-\lambda)\left((4-\lambda)(6-\lambda) - 8\right)$$

$$= (2-\lambda)\left(24 - 10\lambda + \lambda^2 - 8\right)$$

both times -1

$$= (2-\lambda)(\lambda^2 - 10\lambda + 16) = (2-\lambda)(\lambda-2)(\lambda-8)$$

$$= (2-\lambda)^2(8-\lambda)$$

$\underline{\lambda_1 = 2}$: $\text{null}(M - 2I) = \text{null}\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_1}$ iff $x + y + z = 0$ iff $\begin{cases} x = -s-t \\ y = s \\ z = t \end{cases}$ $\therefore B_{\lambda_1} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda_2 = 8}$: $\text{null}(M - 8I) = \text{null}\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -2 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$$= \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V_{\lambda_2}$ iff $\begin{cases} x - z = 0 \\ y - z = 0 \end{cases}$ iff $\begin{cases} x = t \\ y = t \\ z = t \end{cases}$ $\therefore B_{\lambda_2} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\therefore E = B_{\lambda_1} \cup B_{\lambda_2} = \left\{ \overset{v_1}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\} \cup \left\{ \overset{v_3}{\underline{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}} \right\}$ is a basis of $\mathbb{R}^3$ consisting of e-vectors.

NB: v_1 and v_2 are both orthogonal to v_3 (i.e. $v_1 \cdot v_3 = 0 = v_2 \cdot v_3$),
but v_1 and v_2 are not orthogonal to eachother (indeed $v_1 \cdot v_2 = 1 \neq 0$).

Fix: Apply GS-process to B_{λ_1}:

$$u_1 = v_1, \quad u_2 = v_2 - \mathrm{proj}_{u_1}(v_2) = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Finally: normalize u_1, u_2, u_3 to obtain columns of Q:

$$|u_1| = \sqrt{2}, \quad |u_2| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{1+1+4}{4}} = \frac{1}{2}\sqrt{6}, \quad |u_3| = \sqrt{3}.$$

Hence $w_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $w_2 = \frac{2}{\sqrt{6}}\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$, $w_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Therefore: $Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Satisfy $Q^T = Q^{-1}$ and $M = QDQ^T$. ☺ $\boxed{\checkmark}$

Theorem: Let M be a real matrix.
The following are equivalent:

① M is orthogonally diagonalizable.
② M has it's eigenspaces mutually orthogonal.
③ $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of M.
④ M is symmetric.

☺

Thanks for your attention throughout this semester
– Chris E